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SYMMETRICAL MULTILEVEL DIVERSITY CODING WITH AN ALL-ACCESS ENCODER

A Thesis

by

NEEHARIKA MARUKALA

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

May 2012

Major Subject: Electrical Engineering

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Approved by:

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ABSTRACT

Symmetrical Multilevel Diversity Coding with an All-Access Encoder. (May 2012)

Neeharika Marukala, B.Tech., Indian Institute of Technology, Madras

Chair of Advisory Committee: Dr. Tie Liu

Symmetrical Multilevel Diversity Coding (SMDC) is a network compression problem for which a simple separate coding strategy known as superposition coding is optimal in terms of achieving the entire admissible rate region. Carefully constructed induction argument along with the classical subset entropy inequality of Han played a key role in proving the optimality. This thesis considers a generalization of SMDC for which, in addition to the randomly accessible encoders, there is also an all-access encoder. It is shown that superposition coding remains optimal in terms of achieving the entire admissible rate region of the problem. Key to our proof is to identify the supporting hyperplanes that define the boundary of the admissible rate region and then build on a generalization of Han's subset inequality. As a special case, the (R_0, R_s) admissible rate region, which captures all possible tradeoffs between the encoding rate, R_0, of the all-access encoder and the sum encoding rate, R_s, of the randomly accessible encoders, is explicitly characterized. To provide explicit proof of the optimality of superposition coding in this case, a new sliding-window subset entropy inequality is introduced and is shown to directly imply the classical subset entropy inequality of Han.

ACKNOWLEDGMENTS

It is a pleasure to thank the many people who made this thesis possible and my stay at A&M a fun filled one.

I am greatly indebted to my advisor, Dr. Tie Liu, for his excellent guidance, patience, encouragement and constant support. With his enthusiasm and efforts to explain concepts clearly and simply, he helped me gain valuable insights. I cherish every discussion and interaction with him which helped me become a better researcher. It has been a great experience to work with him.

I would like to express my gratitude to Dr. Jean Francois Chamberland for taking time to provide feedback on the thesis.

I consider myself lucky to have had wonderful lab mates. I wish to especially thank Jinjing Jiang, without whom this work would have been incomplete. Hung Ly has been a constant source of motivation, and the many interesting discussions we had further helped my understanding of key concepts.

My stay at Texas A&M has been enjoyable in large part due to the many friends I made here. They have now become a part of my life. I wish to thank all my friends for the entertainment, love and care. I am grateful to Arvind for always being the "friend in need" and for making my stay here a cherishable one.

I would like to thank my uncles, aunts, cousins and undergrad friends for providing me a loving environment.

Lastly, and most importantly, I would like to thank my parents, my husband-to-be Satyan and my brother for their ceaseless love, care and support, which helped me through the years and gave me the strength to reach this point.

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CHAPTER I

INTRODUCTION

Diversity coding (DC) is a network compression problem. In a diversity coding system (DCS), the information is encoded by a number of encoders. However, the decoder has access only to a subset of encoders and has to reconstruct the information either perfectly or subject to some distortion criteria. This is essentially an erasure channel problem for which a maximum distance separable code [9] is to be employed.

DC has a wide range of applications.

1. It is implemented in disk arrays as a fault tolerant measure [7], [6]. Suppose we need to store some information and we store it on a single disk array. In case the disk array breaks down, we have no way of retrieving the information. However, unlike other commodities, information can be broken into pieces. Hence we can employ DC across these pieces and store them on different disk arrays. As a result, when some of the disks are damaged, we still can recover the information from the remaining disks.

2. In computer networks, it improves the reliability of a communication link [1]. When packets are lost owing to link failure or buffer overflow or false routing, the information can be recovered by means of diversity coding. This scheme divides the information in a packet into several pieces and each piece is routed to the destination via a different path. This makes sure the packet can be recovered even when some pieces are lost.

3. It also has applications in secret sharing [5]. In this setting, we have a group of

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This thesis follows the style of *IEEE Transactions on Automatic Control.*

people and each person has an encoded version of a secret. The group can decode the secret only when certain number of people share their parts of information.

Symmetrical Multilevel Diversity Coding (SMDC) is a special kind of diversity coding problem introduced by Roche [7] and Yeung [10]. In this setting, there are a total of L *independent* discrete memoryless sources $(S_1, \ldots, S_L)$, where the importance of the source S_l is assumed to decrease with the subscript l. The sources are to be encoded by a total of L *randomly accessible* encoders. The goal of encoding is to ensure that the number of sources that can be nearly perfectly reconstructed grows with the number of accessible encoders. More specifically, denote by $U \subseteq \Omega_L := \{1, \ldots, L\}$ the set of accessible encoders. The realization of U is *unknown* a priori at the encoders. However, the sources $(S_1, \ldots, S_\alpha)$ need to be nearly perfectly reconstructed whenever $|U| \geq \alpha$. The word "symmetrical" here refers to the fact that the sources that need to be nearly perfectly reconstructed depend on the set of accessible encoders only via its cardinality. The rate allocations at different encoders, however, can be different and are not necessarily symmetrical. It has applications in a distributed storage system wherein the information sources to be stored have an importance order. For example, in binary representation, the most significant bits are literally more important than the least significant bits and in digital imagery, the low frequency components are more important than the high frequency components.

A natural strategy for SMDC is to encode the sources separately at each of the encoders (no coding across different sources) known as *superposition coding* [10]. To show that the natural superposition coding strategy is also optimal, however, turned out to be rather nontrivial. The optimality of superposition coding in terms of achieving the *minimum sum rate* was established by Roche, Yeung, and Hau [8]. The proof used a carefully constructed induction argument, for which the classical subset entropy inequality of Han [2, Ch. 17.6] played a key role. Later, the optimality of superposition coding in terms of achieving the *entire admission rate region* was established by Yeung and Zhang [11]. Their proof was based on a *generalization* of Han's subset inequality, which was established by carefully combining Han's subset inequality with several highly technical results on the analysis of a sequence of linear programs.

In this thesis, we consider a generalization of SMDC for which, in addition to the randomly accessible encoders, there is also an *all-access* encoder. More specifically, in this new setting, a total of $L + 1$ independent discrete memoryless sources $(S_0, S_1, \ldots, S_L)$ are to be encoded by a total of $L+1$ encoders. While encoders 1 to L are randomly accessible encoders as before, encoder 0 is assumed to be an *all-access* encoder. Mathematically, if we denote by $U \subseteq \Omega_L$ the set of randomly accessible encoders whose outputs are actually available at the decoder, then the set of accessible encoders at the decoder is $\{0\} \cup U$. As before, the realization of U is *unknown* a priori at the encoders. However, the sources $(S_0, S_1, \ldots, S_\alpha)$ need to be nearly perfectly reconstructed whenever $|U| \geq \alpha$.

Note that in the above setting, the source S_0 needs to be nearly perfectly reconstructed whenever encoder 0 is accessible. By our assumption, encoder 0 is an all-access encoder. Hence, to minimize the encoding rates, there is *no* need to encode the source S_0 using any of the randomly accessible encoders. If the encoding rate at encoder 0 is set to be the entropy rate of the source S_0, then the sources $(S_1, \ldots, S_L)$ must be encoded by the randomly accessible encoders 1 to L. In this case, the problem reduces to the original setting of Roche [7] and Yeung [10], for which superposition coding is known to be optimal [8, 11]. The main issue that we are concerned with is whether superposition coding will remain optimal when the encoding rate of the all-access encoder 0 is *greater* than the entropy rate of the source S_0.

The main result of the thesis is that superposition coding remains optimal in terms of achieving the entire admissible rate region of SMDC even with the addition of an all-access encoder. Key to our proof is to identify the supporting hyperplanes that define the boundary of the admissible rate region and then builds on the result of Yeung and Zhang on the generalization of Han's subset inequality. As a special case, the (R_0, R_s) admissible rate region, which captures all possible tradeoffs between the encoding rate R_0 of the all-access encoder and the sum encoding rate R_s of the randomly accessible encoders, is explicitly characterized. To provide an explicit proof of the optimality of superposition coding, a *new* sliding-window subset entropy inequality is introduced and is shown to directly imply the classical subset entropy inequality of Han.

The rest of the thesis is organized as follows. The formal statement of the problem and the main result of the thesis are summarized in Chapter II. A proof of the main result is provided in Chapter III. Discussions on the (R_0, R_s) admissible rate region and the sliding-window subset entropy inequality are provided in Chapter IV. Finally, in Chapter V we conclude the thesis with some remarks.

CHAPTER II

PROBLEM STATEMENT AND MAIN RESULT

A. Problem Statement

As illustrated in Figure 1, the problem of SMDC with an all-access encoder consists of:

- a total of $L+1$ *independent* discrete memoryless sources $\{S_\alpha^t\}_{t=1}^\infty$, where $\alpha = 0, 1, \ldots, L$ and t is the time index;
- a set of $L+1$ encoders (encoders 0 to L);
- a decoder who has access to a subset $\{0\} \cup U$ of the encoder outputs, where $U \subseteq \Omega_L$.

The realization of U is *unknown* a priori at the encoders. However, no matter which U actually materializes, the decoder needs to nearly perfectly reconstruct the sources $(S_0, S_1, \ldots, S_\alpha)$ whenever $|U| \geq \alpha$.

Formally, an $(n, (M_0, M_1, \ldots, M_L))$ code is defined by a collection of $L+1$ encoding functions

$$e_l : \prod_{\alpha=0}^{L} \mathcal{S}_\alpha^n \to \{1, \ldots, M_l\}, \quad \forall l = 0, 1, \ldots, L \tag{2.1}$$

and 2^L decoding functions

$$d_U : \{1, \ldots, M_0\} \times \prod_{l \in U} \{1, \ldots, M_l\} \to \prod_{\alpha=0}^{|U|} \mathcal{S}_\alpha^n, \quad \forall U \subseteq \Omega_L. \tag{2.2}$$

A nonnegative rate tuple $(R_0, R_1, \ldots, R_L)$ is said to be *admissible* if for every $\epsilon > 0$, there exits, for sufficiently large block length n, an $(n, (M_0, M_1, \ldots, M_L))$ code such that:

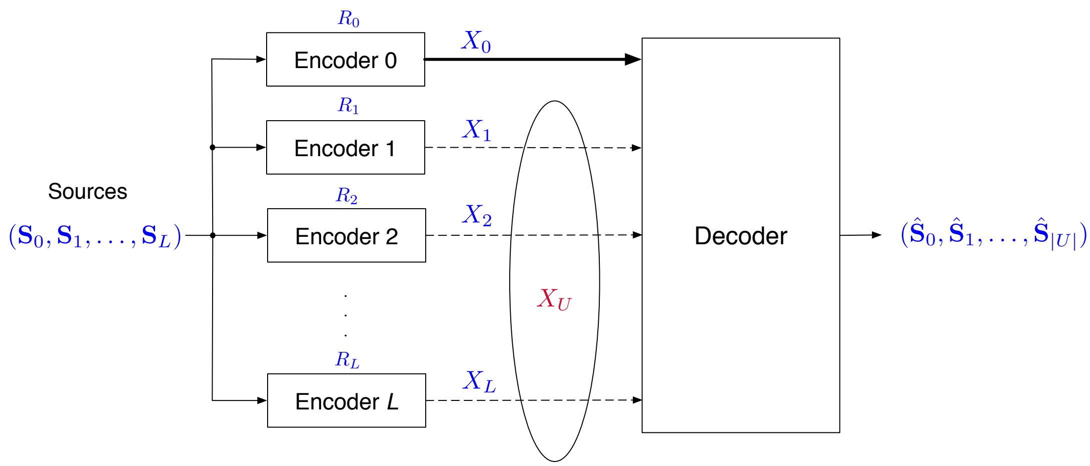

Fig. 1. SMDC with an all-access encoder 0 and L randomly accessible encoders 1 to L. A total of $L+1$ independent discrete memoryless sources $(S_0, S_1, \dots, S_L)$ are to be encoded at the encoders. The decoder, which has access to encoder 0 and a subset U of the randomly accessible encoders, needs to nearly perfectly reconstruct the sources $(S_0, S_1, \dots, S_{|U|})$ no matter what the realization of U is.

- (Rate constraints at the encoders)

$$\frac{1}{n} \log M_l \leq R_l + \epsilon, \qquad \forall l = 0, 1, \dots, L; \tag{2.3}$$

- (Asymptotically perfect reconstructions at the decoder)

$$\Pr\left\{d_U(X_0, X_U) \neq (\mathbf{S}_0, \mathbf{S}_1, \dots, \mathbf{S}_{|U|})\right\} \leq \epsilon, \qquad \forall U \subseteq \Omega_L \tag{2.4}$$

where $\mathbf{S}_\alpha := \{S_\alpha^n\}_{t=1}^n$, $X_l = e_l(\mathbf{S}_0, \mathbf{S}_1, \dots, \mathbf{S}_L)$ is the output of encoder l, and $X_U := \{X_l : l \in U\}$.

The *admissible rate region* $\mathcal{R}$ is the collection of *all* admissible rate tuples $(R_0, R_1, \dots, R_L)$.

B. Superposition Coding Rate Region

As mentioned previously, a natural strategy for SMDC is superposition coding, i.e., to encode the sources separately at the encoders and there is no coding across different sources. Formally, the problem of encoding a single source S_α can be viewed as a special case of the general SMDC problem where the sources S_m are constants for all $m \neq \alpha$. In this case, the source S_α needs to be nearly perfectly reconstructed whenever the decoder can access at least α randomly accessible encoders in addition to the all-access encoder. Thus, the problem is essentially to transmit S_α over an *erasure* channel, and the following simple source-channel separation scheme is known to be optimal (whether the all-access encoder exists or not) [7, 10]:

- First compress the source sequence $\mathbf{S}_\alpha$ into a source message W using a *lossless* source code. It is well known [2, Ch. 5] that the rate of the source message W can be made arbitrarily close to the entropy rate $H(S_\alpha)$ for sufficiently large block length n.

- Next, the source message W is encoded at encoders 0 to L using a *maximum distance separable* code [9]. It is well known [7, 10] that the source message W can be perfectly recovered at the decoder whenever

$$R_0 + \sum_{l \in U} R_l \geq H(S_\alpha), \quad \forall U \in \Omega_L^{(\alpha)} \tag{2.5}$$

  where $\Omega_L^{(\alpha)}$ is the collection of all subsets of Ω_L of size α.

We summarize the above result into the following proposition.

Proposition 1. *The admissible rate region for encoding a single source S_α is given by the collection of all nonnegative rate tuples $(R_0, R_1, \ldots, R_L)$ satisfying (2.5).*

By Proposition 1, the superposition coding rate region $\mathcal{R}_{sup}$ for SMDC with an all-access encoder is given by the collection of all nonnegative rate tuples $(R_0, R_1, \ldots, R_L)$ such that

$$R_l = \sum_{\alpha=0}^{L} r_l^{(\alpha)} \tag{2.6}$$

for some nonnegative $r_l^{(\alpha)}$, $\alpha = 0, 1, \ldots, L$ and $l = 0, 1, \ldots, L$, satisfying

$$r_0^{(\alpha)} + \sum_{l \in U} r_l^{(\alpha)} \geq H(S_\alpha), \quad \forall U \in \Omega_L^{(\alpha)}. \tag{2.7}$$

Note that in theory, an explicit characterization of the superposition coding rate region $\mathcal{R}_{sup}$ can be obtained by eliminating $r_l^{(\alpha)}$, $\alpha = 0, 1, \ldots, L$ and $l = 0, 1, \ldots, L$, via a Fourier-Motzkin elimination from (2.6) and (2.7). The elimination process, however, becomes *unmanageable* even for moderate L, as there are simply too many equations involved.

C. Main Result

The main result of the thesis is that superposition coding can achieve the entire admissible rate region $\mathcal{R}$ of SMDC with an all-access encoder, as summarized in the following theorem.

Theorem 1.

$$\mathcal{R} = \mathcal{R}_{sup} \tag{2.8}$$

A detailed proof of the theorem is provided in Chapter III. Below, we summarize the main technical ingredients of the proof. First, our proof relies on the following characterization of the superposition coding rate region R_{sup}. Let $(\lambda_1, \ldots, \lambda_L)$ be a nonnegative vector in $\mathbb{R}^L$ and let f_α be the *optimal* value of the linear program

$$\begin{array}{rl} \max & \sum_{U \in \Omega_L^{(\alpha)}} c_\alpha(U) \\ \text{subject to} & \sum_{U \in \Omega_L^{(\alpha)}, U \ni l} c_\alpha(U) \leq \lambda_l, \quad \forall l = 1, \ldots, L \\ & c_\alpha(U) \geq 0, \quad \forall U \in \Omega_L^{(\alpha)} \end{array} \tag{2.9}$$

for $\alpha = 1, \ldots, L$. Denote by $\mathcal{R}^*$ the collection of all nonnegative rate tuples $(R_0, R_1, \ldots, R_L)$ satisfying

$$R_0 \geq H(S_0) \tag{2.10}$$

and

$$f_m R_0 + \sum_{l=1}^{L} \lambda_l R_l \geq f_m \sum_{\alpha=0}^{m} H(S_\alpha) + \sum_{\alpha=m+1}^{L} f_\alpha H(S_\alpha) \tag{2.11}$$

for all $m = 1, \ldots, L$ and all nonnegative $(\lambda_1, \ldots, \lambda_L)$ in $\mathbb{R}^L$.

Proposition 2.

$$\mathcal{R}_{sup} = \mathcal{R}^*. \tag{2.12}$$

A proof of the proposition is provided in Chapter III, Section A. The proof uses the fact that $\mathcal{R}_{sup}$ is a *polyhedron* with polyhedral cone being the nonnegative orthant in $\mathbb{R}^{L+1}$ and hence can be completely characterized by the supporting hyperplanes

$$\sum_{l=0}^{L} \lambda_l R_l \geq f, \quad \forall (\lambda_0, \lambda_1, \ldots, \lambda_L) \geq 0 \tag{2.13}$$

where

$$f = \min_{(R_0, R_1, \ldots, R_L) \in \mathcal{R}_{sup}} \sum_{l=0}^{L} \lambda_l R_l. \tag{2.14}$$

We then complete the proof by showing that for any nonzero $(\lambda_1, \ldots, \lambda_L)$, the faces of $\mathcal{R}_{sup}$ are only determined by the supporting hyperplanes with $\lambda_0 = f_m$ for $m = 1, \ldots, L$.

The second key ingredient of our proof is the following generalization of the classical subset entropy inequality of Han, which was first established in [11, Theorem 3].

Proposition 3 (Generalized Han's subset inequality)**.** *For any nonnegative* $(\lambda_1, \dots, \lambda_L)$ *in* $\mathbb{R}^L$*, there exists a set of* $c_\alpha := \{c_\alpha(U) : U \in \Omega_L^{(\alpha)}\}$*,* $\alpha = 1, \dots, L$*, for which each* c_α *is an optimal solution to the linear program* (2.9) *and such that*

$$\sum_{U \in \Omega_L^{(1)}} c_1(U)H(X_U) \geq \sum_{U \in \Omega_L^{(2)}} c_2(U)H(X_U) \geq \cdots \geq \sum_{U \in \Omega_L^{(L)}} c_L(U)H(X_U) \tag{2.15}$$

for any collection of L *jointly distributed random variables* $(X_1, \dots, X_L)$*.*

In [11], Proposition 3 was established via a *delicate* combination of Han's subset inequality and several highly technical results on the analysis of the linear program (2.9). In Chapter III, Section B, we provide a more structured proof which completely separates the entropy argument from the analysis of the linear program (2.9). The proof is based on a subset entropy inequality recently established by Madiman and Tetali [3] and an analysis result on the linear program (2.9) *lifted* from the original proof of Yeung and Zhang [11].

The following corollaries will be directly used in the proof of Theorem 1.

Corollary 1. *For any nonnegative* $(\lambda_1, \dots, \lambda_L)$ *in* $\mathbb{R}^L$*,*

$$f_1 \geq 2f_2 \geq \cdots \geq Lf_L. \tag{2.16}$$

In particular,

$$f_1 \geq f_2 \geq \cdots \geq f_L. \tag{2.17}$$

Proof. Let c_α, $\alpha = 1, \dots, L$, be a set of optimal solutions to the linear program (2.9) such that the inequality chain (2.15) holds for any collection of L jointly distributed random variables $(X_1, \dots, X_L)$. In particular, let $(X_1, \dots, X_L)$ be a collection of L *independent and identically distributed* random variables. We have $H(X_U) = \alpha H(X_1)$

for all $U \in \Omega_L^{(\alpha)}$ and

$$\sum_{U \in \Omega_L^{(\alpha)}} c_\alpha(U) H(X_U) = \alpha H(X_1) \sum_{U \in \Omega_L^{(\alpha)}} c_\alpha(U) = \alpha f_\alpha H(X_1) \tag{2.18}$$

for all $\alpha = 1, \ldots, L$. Here, the second equality is due to the fact that c_α is optimal so $\sum_{U \in \Omega_L^{(\alpha)}} c_\alpha(U) = f_\alpha$. Substituting (2.18) into the inequality chain (2.15) and dividing each term by $H(X_1)$ give the inequality chain (2.16).

For any $\alpha = 2, \ldots, L$, note from the inequality chain (2.16) that

$$f_{\alpha-1} \geq \frac{\alpha}{\alpha - 1} f_\alpha \geq f_\alpha. \tag{2.19}$$

This proves the inequality chain (2.17). □

Corollary 2. *For any nonnegative* $(\lambda_1, \ldots, \lambda_L)$ *in* $\mathbb{R}^L$*, there exists a set of* c_α*,* $\alpha = 1, \ldots, L$*, for which each* c_α *is an optimal solution to the linear program* (2.9) *and such that*

$$\sum_{U \in \Omega_L^{(1)}} c_1(U) H(X_0, X_U | T) \geq \sum_{U \in \Omega_L^{(2)}} c_2(U) H(X_0, X_U | T) \geq \cdots \geq \sum_{U \in \Omega_L^{(L)}} c_L(U) H(X_0, X_U | T) \tag{2.20}$$

for any collection of $L + 2$ *jointly distributed random variables* $(X_0, X_1, \ldots, X_L, T)$*.*

Proof. Given $T = t$, apply Proposition 3 with the vector-valued jointly distributed random variables $((X_0, X_1), \ldots, (X_0, X_L))$. We have

$$\sum_{U \in \Omega_L^{(1)}} c_1(U) H(X_0, X_U | T = t) \geq \sum_{U \in \Omega_L^{(2)}} c_2(U) H(X_0, X_U | T = t) \geq \cdots \geq \sum_{U \in \Omega_L^{(L)}} c_L(U) H(X_0, X_U | T = t). \tag{2.21}$$

Averaging over t completes the proof of the corollary. □

CHAPTER III

PROOF OF THE MAIN RESULT

A. Proof of Proposition 2

First note that the superposition coding rate region $\mathcal{R}_{sup}$ as described by (2.6) and (2.7) is a polyhedron with polyhedral cone being the nonnegative orthant in $\mathbb{R}^{L+1}$, so we can write $\mathcal{R}_{sup}$ as the collection of the nonnegative rate tuples $(R_0, R_1, \ldots, R_L)$ such that

$$\sum_{l=0}^{L} \lambda_l R_l \geq f \tag{3.1}$$

for all $(\lambda_0, \lambda_1, \ldots, \lambda_L) \geq 0$, where

$$f = \min_{(R_0,R_1,\ldots,R_L)\in\mathcal{R}_{sup}} \sum_{l=0}^{L} \lambda_l R_l. \tag{3.2}$$

Substituting (2.6) into (3.2), we can write the objective function of the optimization problem as

$$\sum_{l=0}^{L} \lambda_l R_l = \sum_{l=0}^{L} \left(\lambda_l \sum_{\alpha=0}^{L} r_l^{(\alpha)} \right) = \sum_{\alpha=0}^{L} \left(\sum_{l=0}^{L} \lambda_l r_l^{(\alpha)} \right). \tag{3.3}$$

Note from (2.7) that the constraints that define the superposition coding rate region $\mathcal{R}_{sup}$ are completely separated for different α. Hence, the optimization problem (3.2) can be solved by solving for each $\alpha = 0, \ldots, L$ the optimization problem

$$\begin{array}{rl} \min & \sum_{l=0}^{L} \lambda_l r_l^{(\alpha)} \\ \text{subject to} & r_0^{(\alpha)} + \sum_{l\in U} r_l^{(\alpha)} \geq H(S_\alpha), \quad \forall U \subseteq \Omega_L^{(\alpha)} \\ & r_l^{(\alpha)} \geq 0, \quad \forall l = 0, 1, \ldots, L. \end{array} \tag{3.4}$$

Let f'_α be the optimal value of the optimization problem (3.4). Clearly, $f'_0 = \lambda_0 H(S_0)$. For $\alpha = 1, \ldots, L$, since the optimization problem (3.4) is linear, by strong duality f'_α is also the optimal value of the dual program

$$\begin{array}{rl} \max & \left(\sum_{U\in\Omega_L^{(\alpha)}} c_\alpha(U)\right) H(S_\alpha) \\ \text{subject to} & \sum_{U\in\Omega_L^{(\alpha)}} c_\alpha(U) \le \lambda_0 \\ & \sum_{U\in\Omega_L^{(\alpha)}, U\ni l} c_\alpha(U) \le \lambda_l, \quad \forall l = 1,\dots,L \\ & c_\alpha(U) \ge 0, \quad \forall U \in \Omega_L^{(\alpha)}. \end{array} \tag{3.5}$$

Note that if the constraint $\sum_{U\in\Omega_L^{(\alpha)}} c_\alpha(U) \le \lambda_0$ is inactive, it can be removed from the dual program (3.5). In this case the optimal value $f'_\alpha = f_\alpha H(S_\alpha)$, where f_α is the optimal value of the linear program (2.9). On the other hand, if the constraint $\sum_{U\in\Omega_L^{(\alpha)}} c_\alpha(U) \le \lambda_0$ is active, the optimal value $f'_\alpha = \lambda_0 H(S_\alpha)$. Combing these two cases, we have

$$f'_\alpha = \min(\lambda_0 H(S_\alpha), f_\alpha H(S_\alpha)) = \min(\lambda_0, f_\alpha) H(S_\alpha) \tag{3.6}$$

for $\alpha = 1,\dots,L$ and hence

$$f = \sum_{\alpha=0}^{L} f'_\alpha = \lambda_0 H(S_0) + \sum_{\alpha=1}^{L} \min(\lambda_0, f_\alpha) H(S_\alpha). \tag{3.7}$$

Substituting (3.7) into (3.1), we conclude that the superposition coding rate region $\mathcal{R}_{sup}$ is given by the collection of the nonnegative rate tuples $(R_0, R_1, \dots, R_L)$ such that

$$\sum_{l=0}^{L} \lambda_l R_l \ge \lambda_0 H(S_0) + \sum_{\alpha=1}^{L} \min(\lambda_0, f_\alpha) H(S_\alpha) \tag{3.8}$$

for all $(\lambda_0, \lambda_1, \dots, \lambda_L) \ge 0$.

To show that $\mathcal{R}_{sup} \subseteq \mathcal{R}^*$, let $(R_0, R_1, \dots, R_L)$ be a nonnegative rate tuple in $\mathcal{R}_{sup}$ that satisfies (3.8) for all $(\lambda_0, \lambda_1, \dots, \lambda_L) \ge 0$. Note that when $(\lambda_1, \dots, \lambda_L) = 0$, $f_\alpha = 0$ for all $\alpha = 1, \dots, L$. Thus, let $(\lambda_0, \lambda_1, \dots, \lambda_L) = (1, 0, \dots, 0)$ and we have from (3.8) that the rate tuple $(R_0, R_1, \dots, R_L)$ must satisfy the inequality (2.10).

Furthermore, for any $m = 1, \ldots, L$, by the inequality chain (2.17) we have

$$\sum_{\alpha=1}^{L} \min(f_m, f_\alpha) H(S_\alpha) = f_m \sum_{\alpha=1}^{m} H(S_\alpha) + \sum_{\alpha=m+1}^{L} f_\alpha H(S_\alpha). \tag{3.9}$$

Thus, let $\lambda_0 = f_m$ and we have from (3.8) and (3.9) that the rate tuple $(R_0, R_1, \ldots, R_L)$ must also satisfy the inequality (2.11) for all $m = 1, \ldots, L$.

This proves that $(R_0, R_1, \ldots, R_L) \in \mathcal{R}^*$ and hence $\mathcal{R}_{sup} \subseteq \mathcal{R}^*$.

To show the reverse relationship $\mathcal{R}^* \subseteq \mathcal{R}_{sup}$, let $(R_0, R_1, \ldots, R_L)$ be a nonnegative rate tuple in $\mathcal{R}^*$ that satisfies (2.10) and (2.11) for all $m = 1, \ldots, L$ and all nonnegative $(\lambda_1, \ldots, \lambda_L)$ in $\mathbb{R}^L$. To show that the rate tuple $(R_0, R_1, \ldots, R_L)$ must also satisfy the inequality (3.8) for all $(\lambda_0, \lambda_1, \ldots, \lambda_L) \geq 0$, let us consider the following three cases separately.

Case 1: $\lambda_0 \geq f_1$. In this case, note that the inequality (2.11) with $m = 1$ can be written as

$$f_1 R_0 + \sum_{l=1}^{L} \lambda_l R_l \geq f_1 H(S_0) + \sum_{\alpha=1}^{L} f_\alpha H(S_\alpha). \tag{3.10}$$

From the inequalities (3.10) and (2.10), we have

$$\begin{aligned} \lambda_0 R_0 + \sum_{l=1}^{L} \lambda_l R_l &= (\lambda_0 - f_1) R_0 + \left(f_1 R_0 + \sum_{l=1}^{L} \lambda_l R_l \right) && (3.11) \\ &\geq (\lambda_0 - f_1) H(S_0) + \left(f_1 H(S_0) + \sum_{\alpha=1}^{L} f_\alpha H(S_\alpha) \right) && (3.12) \\ &= \lambda_0 H(S_0) + \sum_{\alpha=1}^{L} f_\alpha H(S_\alpha) && (3.13) \end{aligned}$$

which is exactly the inequality (3.8) with $\lambda_0 \geq f_1$.

Case 2: $\lambda_0 < f_L$. In this case, note that the inequality (2.11) with $m = L$ can be written as

$$f_L R_0 + \sum_{l=1}^{L} \lambda_l R_l \geq f_L \sum_{\alpha=0}^{L} H(S_\alpha). \tag{3.14}$$

From the inequality (3.14), we have

$$\begin{aligned}
\lambda_0 R_0 + \sum_{l=1}^{L} \lambda_l R_l &= \frac{\lambda_0}{f_L}\left(f_L R_0 + \sum_{l=1}^{L} \lambda_l R_l\right) + \left(1 - \frac{\lambda_0}{f_L}\right) \sum_{l=1}^{L} \lambda_l R_l \qquad (3.15)\\
&\geq \frac{\lambda_0}{f_L}\left(f_L R_0 + \sum_{l=1}^{L} \lambda_l R_l\right) \qquad (3.16)\\
&\geq \frac{\lambda_0}{f_L}\left(f_L \sum_{\alpha=0}^{L} H(S_\alpha)\right) \qquad (3.17)\\
&= \lambda_0 \sum_{\alpha=0}^{L} H(S_\alpha) \qquad (3.18)
\end{aligned}$$

which is exactly the inequality (3.8) with $\lambda_0 < f_L$.

Case 3: $\lambda_0 \in [f_{r+1}, f_r)$ for some integer r between 1 and $L-1$. In this case, note that the inequality (2.11) with $m = r$ and $m = r+1$ can be written as

$$\begin{aligned}
f_r R_0 + \sum_{l=1}^{L} \lambda_l R_l &\geq f_r \sum_{\alpha=0}^{r} H(S_\alpha) + \sum_{\alpha=r+1}^{L} f_\alpha H(S_\alpha) \qquad (3.19)\\
\text{and} \quad f_{r+1} R_0 + \sum_{l=1}^{L} \lambda_l R_l &\geq f_{r+1} \sum_{\alpha=0}^{r+1} H(S_\alpha) + \sum_{\alpha=r+2}^{L} f_\alpha H(S_\alpha) \qquad (3.20)
\end{aligned}$$

respectively. From the inequalities (3.19) and (3.20), we have

$$\begin{aligned}
\lambda_0 R_0 + \sum_{l=1}^{L} \lambda_l R_l &= \frac{\lambda_0 - f_{r+1}}{f_r - f_{r+1}}\left(f_r R_0 + \sum_{l=1}^{L} \lambda_l R_l\right) + \\
&\qquad \frac{f_r - \lambda_0}{f_r - f_{r+1}}\left(f_{r+1} R_0 + \sum_{l=1}^{L} \lambda_l R_l\right) \qquad (3.21)\\
&\geq \frac{\lambda_0 - f_{r+1}}{f_r - f_{r+1}}\left(f_r \sum_{\alpha=0}^{r} H(S_\alpha) + \sum_{\alpha=r+1}^{L} f_\alpha H(S_\alpha)\right) + \\
&\qquad \frac{f_r - \lambda_0}{f_r - f_{r+1}}\left(f_{r+1} \sum_{\alpha=0}^{r+1} H(S_\alpha) + \sum_{\alpha=r+2}^{L} f_\alpha H(S_\alpha)\right) \qquad (3.22)\\
&= \lambda_0 \sum_{\alpha=0}^{r} H(S_\alpha) + \sum_{\alpha=r+1}^{L} f_\alpha H(S_\alpha) \qquad (3.23)
\end{aligned}$$

which is exactly the inequality (3.8) with $\lambda_0 \in [f_{r+1}, f_r)$.

Combining these three cases proves that the rate tuple $(R_0, R_1, \ldots, R_L) \in \mathcal{R}_{sup}$ and hence $\mathcal{R}^* \subseteq \mathcal{R}_{sup}$. Combined with the previous result $\mathcal{R}_{sup} \subseteq \mathcal{R}^*$, we conclude that $\mathcal{R}_{sup} = \mathcal{R}^*$. This completes the proof of Proposition 2.

B. Proof of Proposition 3

Note that it is sufficient to prove that for any nonnegative $(\lambda_1, \ldots, \lambda_L)$ in $\mathbb{R}^L$, any $\alpha = 2, \ldots, L$, and any c_α which is an optimal solution to the linear program (2.9), there exists a $c_{\alpha-1}$ which is also optimal for the linear program

$$\begin{array}{rl} \max & \sum_{V \in \Omega_L^{(\alpha-1)}} c_{\alpha-1}(V) \\ \text{subject to} & \sum_{V \in \Omega_L^{(\alpha-1)}, V \ni l} c_{\alpha-1}(V) \leq \lambda_l, \quad \forall l = 1, \ldots, L \\ & c_{\alpha-1}(V) \geq 0, \quad \forall V \in \Omega_L^{(\alpha-1)} \end{array} \tag{3.24}$$

and such that

$$\sum_{V \in \Omega_L^{(\alpha-1)}} c_{\alpha-1}(V) H(X_V) \geq \sum_{U \in \Omega_L^{(\alpha)}} c_\alpha(U) H(X_U) \tag{3.25}$$

for any collection of L jointly distributed random variables $(X_1, \ldots, X_L)$. To prove the existence of such a $c_{\alpha-1}$, let us recall the following two results from the literature.

First, a subset entropy inequality from [3, Theorem I'] which we rephrase here as follows. Consider a *hypergraph* $(U, \mathcal{V})$ where $U \subseteq \Omega_L$ is a finite ground set and $\mathcal{V}$ is a collection of subsets of U. A function $g : \mathcal{V} \to \mathbb{R}^+$ is called a *fractional cover* of $(U, \mathcal{V})$ if it satisfies

$$\sum_{V \in \mathcal{V}, V \ni i} g(V) \geq 1, \quad \forall i \in U. \tag{3.26}$$

Lemma 1 (A subset entropy inequality of Madiman and Tetali). *Let $(U, \mathcal{V})$ be a hypergraph where the ground set $U \subseteq \Omega_L$, and let g be a fractional cover of $(U, \mathcal{V})$.*

Then

$$H(X_U) \le \sum_{V \in \mathcal{V}} g(V) H(X_V) \tag{3.27}$$

for any collection of L jointly distributed random variables $(X_1, \ldots, X_L)$.

The second one is an analysis result on the linear program (2.9), which we lift from the proof of [11, Theorem 3]. Consider the hypergraph $(U, \mathcal{V}_U)$ where $U \in \Omega_L^{(\alpha)}$ and $\mathcal{V}_U := \{V \in \Omega_L^{(\alpha-1)} : V \subseteq U\}$. Let g_U be a fractional cover of $(U, \mathcal{V}_U)$ and let $\mathcal{G}_L^{(\alpha)} := \{g_U : U \in \Omega_L^{(\alpha)}\}$.

Lemma 2. *For any nonnegative $(\lambda_1, \ldots, \lambda_L)$ in $\mathbb{R}^L$, any $\alpha = 2, \ldots, L$, and any c_α which is an optimal solution to the linear program (2.9) with the optimal value $f_\alpha > 0$, there exists a set of fractional covers $\mathcal{G}_L^{(\alpha)}$ such that $c_{\alpha-1} = \{c_{\alpha-1}(V) : V \in \Omega_L^{(\alpha-1)}\}$ where*

$$c_{\alpha-1}(V) = \sum_{U \in \Omega_L^{(\alpha)}, U \supseteq V} c_\alpha(U) g_U(V) \tag{3.28}$$

is an optimal solution to the linear program (3.24).

A proof of Lemma 2 is provided in Appendix A. When $f_\alpha = 0$, $c_\alpha(U) = 0$ for all $U \in \Omega_L^{(\alpha)}$. Thus, any $c_{\alpha-1}$ which is optimal to the linear program (3.24) trivially satisfies

$$\sum_{V \in \Omega_L^{(\alpha-1)}} c_{\alpha-1}(V) H(X_V) \ge \sum_{U \in \Omega_L^{(\alpha)}} c_\alpha(U) H(X_U) = 0. \tag{3.29}$$

When $f_\alpha > 0$, combining (3.27) and (3.28) gives

$$\sum_{U \in \Omega_L^{(\alpha)}} c_\alpha(U) H(X_U) \leq \sum_{U \in \Omega_L^{(\alpha)}} c_\alpha(U) \left(\sum_{V \in \mathcal{V}_U^{(\alpha-1)}} g_U(V) H(X_V) \right) \tag{3.30}$$

$$= \sum_{V \in \Omega_L^{(\alpha-1)}} \left(\sum_{U \in \Omega_L^{(\alpha)}, U \supseteq V} c_\alpha(U) g_U(V) \right) H(X_V) \tag{3.31}$$

$$= \sum_{V \in \Omega_L^{(\alpha-1)}} c_{\alpha-1}(V) H(X_V) \tag{3.32}$$

for any collection of L jointly distributed random variables $(X_1, \ldots, X_L)$. This completes the proof of Proposition 3.

C. Proof of Theorem 1

Since we naturally have $\mathcal{R}_{sup} \subseteq \mathcal{R}$, to show that $\mathcal{R}_{sup} = \mathcal{R}$, it is sufficient to show that $\mathcal{R} \subseteq \mathcal{R}_{sup}$. To show that $\mathcal{R} \subseteq \mathcal{R}_{sup}$, we need to show that *any* admissible rate tuple $(R_0, R_1, \ldots, R_L)$ must satisfy the inequalities (2.10) and (2.11) for all $m = 1, \ldots, L$ and all nonnegative $(\lambda_1, \ldots, \lambda_L)$ in $\mathbb{R}^L$. The fact that any admissible rate tuple $(R_0, R_1, \ldots, R_L)$ must satisfy the inequality (2.10) follows directly from Proposition 1 for encoding the single source S_0.

To show that any admissible rate tuple $(R_0, R_1, \ldots, R_L)$ must satisfy the inequality (2.11) for all $m = 1, \ldots, L$ and all nonnegative $(\lambda_1, \ldots, \lambda_L)$ in $\mathbb{R}^L$, let c_α, $\alpha = 1, \ldots, L$, be a set of optimal solutions for the linear program (2.9) such that the inequality chain (2.20) holds for any collection of $L+2$ jointly distributed random variables $(X_0, X_1, \ldots, X_L, T)$. Note that for $\alpha = 1$, the optimal solution for the linear program (2.9) is *unique* and is given by

$$c_1(\{l\}) = \lambda_l, \quad \forall l = 1, \ldots, L. \tag{3.33}$$

We thus have

$$
\begin{aligned}
n\left(f_m R_0 + \sum_{l=1}^{L} \lambda_l R_l\right) &= f_m n R_0 + \sum_{l=1}^{L} c_1(\{l\}) n R_l && (3.34)\\
&\geq f_m(H(X_0) - n\epsilon) + \sum_{l=1}^{L} c_1(\{l\})(H(X_l) - n\epsilon) && (3.35)\\
&= f_m H(X_0) + \sum_{l=1}^{L} c_1(\{l\}) H(X_l) - n(f_1 + f_m)\epsilon && (3.36)\\
&\geq f_m H(X_0) + \sum_{U \in \Omega_L^{(m)}} c_m(U) H(X_U) - n(f_1 + f_m)\epsilon && (3.37)\\
&= \sum_{U \in \Omega_L^{(m)}} c_m(U)(H(X_0) + H(X_U)) - n(f_1 + f_m)\epsilon && (3.38)\\
&\geq \sum_{U \in \Omega_L^{(m)}} c_m(U) H(X_0, X_U) - n(f_1 + f_m)\epsilon && (3.39)
\end{aligned}
$$

where (3.35) follows from the rate constraint (2.3), (3.36) follows from the fact that c_1 is an optimal solution so $f_1 = \sum_{l=1}^{L} c_1(\{l\})$, (3.37) follows from the fact that

$$
\sum_{l=1}^{L} c_1(\{l\}) H(X_l) \geq \sum_{U \in \Omega_L^{(m)}} c_m(U) H(X_U) \tag{3.40}
$$

as specified by the inequality chain (2.20), (3.38) follows from the fact that c_m is an optimal solution so $f_m = \sum_{U \in \Omega_L^{(m)}} c_m(U)$, and (3.39) follows from the independence bound on entropy.

For any $U \in \Omega_L^{(m)}$, by the asymptotically perfect reconstruction constraint (2.4) and Fano's inequality we have

$$
H(\mathbf{S}_0^m | X_0, X_U) \leq 1 + n\epsilon \sum_{\alpha=0}^{m} \log |\mathcal{S}_\alpha|. \tag{3.41}
$$

where $\mathbf{S}_0^m$ denotes the random variables $\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_m$. By the chain rule for entropy, we have

$$
\begin{aligned}
H(X_0, X_U) &= H(X_0, X_U, \mathbf{S}_0^m) - H(\mathbf{S}_0^m | X_0, X_U) && (3.42)\\
&= H(\mathbf{S}_0^m) + H(X_0, X_U | \mathbf{S}_0^m) - H(\mathbf{S}_0^m | X_0, X_U) && (3.43)\\
&= n \sum_{\alpha=0}^{m} H(S_\alpha) + H(X_0, X_U | \mathbf{S}_0^m) - H(\mathbf{S}_0^m | X_0, X_U) && (3.44)\\
&\geq n \sum_{\alpha=0}^{m} H(S_\alpha) + H(X_0, X_U | \mathbf{S}_0^m) - \left(1 + n\epsilon \sum_{\alpha=0}^{m} \log |\mathcal{S}_\alpha|\right) && (3.45)
\end{aligned}
$$

where (3.44) is due to the fact that $S_0, S_1, \ldots, S_L$ are independent memoryless sources, and (3.45) follows from (3.41). Substituting (3.45) into (3.39), we have

$$
\begin{aligned}
n\left(f_m R_0 + \sum_{l=1}^{L} \lambda_l R_l\right) &\geq \sum_{U \in \Omega_L^{(m)}} c_m(U) \left(n \sum_{\alpha=0}^{m} H(S_\alpha) + H(X_0, X_U | \mathbf{S}_0^m) - \right.\\
&\qquad \left. \left(1 + n\epsilon \sum_{\alpha=0}^{m} \log |\mathcal{S}_\alpha|\right)\right) - n(f_1 + f_m)\epsilon && (3.46)\\
&= n f_m \sum_{\alpha=0}^{m} H(S_\alpha) + \sum_{U \in \Omega_L^{(m)}} c_m(U) H(X_0, X_U | \mathbf{S}_0^m) - \\
&\qquad \left(f_m + n\epsilon \left(f_m \sum_{\alpha=0}^{m} \log |\mathcal{S}_\alpha| + f_1 + f_m\right)\right). && (3.47)
\end{aligned}
$$

To proceed, let us show, via an induction, that for any admissible code with encoder outputs $\{X_l\}_{l=0}^{L}$, any nonnegative $(\lambda_1, \ldots, \lambda_L)$ in $\mathbb{R}^L$, and any $m = 1, \ldots, L$, we have

$$
\sum_{U \in \Omega_L^{(m)}} c_m(U) H(X_0, X_U | \mathbf{S}_0^m) \geq n \sum_{\alpha=m+1}^{L} f_\alpha H(S_\alpha) - \sum_{\alpha=m+1}^{L} f_\alpha \left(1 + n\epsilon \log |\mathcal{S}_\alpha|\right). \qquad (3.48)
$$

First, when $m = L$, the inequality (3.48) is trivial as the right-hand side of the inequality is zero. Next, assume that the inequality (3.48) holds for some $m = l$, i.e,

$$
\sum_{U \in \Omega_L^{(l)}} c_l(U) H(X_0, X_U | \mathbf{S}_0^l) \geq n \sum_{\alpha=l+1}^{L} f_\alpha H(S_\alpha) - \sum_{\alpha=l+1}^{L} f_\alpha \left(1 + n\epsilon \log |\mathcal{S}_\alpha|\right). \qquad (3.49)
$$

For any $U \in \Omega_L^{(l)}$, we have

$$\begin{aligned} H(X_0, X_U|\mathbf{S}_0^l) &= H(X_0, X_U|\mathbf{S}_0^{l-1}) - I(\mathbf{S}_l; X_0, X_U|\mathbf{S}_0^{l-1}) && (3.50)\\ &= H(X_0, X_U|\mathbf{S}_0^{l-1}) - H(\mathbf{S}_l|\mathbf{S}_0^{l-1}) + H(\mathbf{S}_l|X_0, X_U, \mathbf{S}_0^{l-1}) && (3.51)\\ &\leq H(X_0, X_U|\mathbf{S}_0^{l-1}) - H(\mathbf{S}_l|\mathbf{S}_0^{l-1}) + H(\mathbf{S}_l|X_0, X_U) && (3.52)\\ &\leq H(X_0, X_U|\mathbf{S}_0^{l-1}) - H(\mathbf{S}_l|\mathbf{S}_0^{l-1}) + (1 + n\epsilon \log|\mathcal{S}_l|) && (3.53)\\ &= H(X_0, X_U|\mathbf{S}_0^{l-1}) - nH(S_l) + (1 + n\epsilon \log|\mathcal{S}_l|) && (3.54) \end{aligned}$$

where (3.52) follows from the fact that conditioning reduces entropy, (3.53) follows the asymptotically perfect reconstruction constraint (2.4) and Fano's inequality so we have

$$H(\mathbf{S}_l|X_0, X_U) \leq 1 + n\epsilon \log|\mathcal{S}_l| \quad (3.55)$$

and (3.54) follows from the fact that $S_0, S_1, \ldots, S_L$ are independent memoryless sources. Multiplying both sides of the inequality (3.54) by $c_l(U)$ and summing over all $U \in \Omega_L^{(l)}$, we have

$$\begin{aligned} &\sum_{U\in\Omega_L^{(l)}} c_l(U) H(X_0, X_U|\mathbf{S}_0^{l-1}) \\ &\quad \geq \sum_{U\in\Omega_L^{(l)}} c_l(U) \left(H(X_0, X_U|\mathbf{S}_0^l) + nH(S_l) - (1 + n\epsilon \log|\mathcal{S}_l|)\right) && (3.56)\\ &\quad = \sum_{U\in\Omega_L^{(l)}} c_l(U) H(X_0, X_U|\mathbf{S}_0^l) + nf_l H(S_l) - f_l\left(1 + n\epsilon \log|\mathcal{S}_l|\right) && (3.57)\\ &\quad \geq n \sum_{\alpha=l+1}^{L} f_\alpha H(S_\alpha) - \sum_{\alpha=l+1}^{L} f_\alpha \left(1 + n\epsilon \log|\mathcal{S}_\alpha|\right) \\ &\qquad\qquad + nf_l H(S_l) - f_l\left(1 + n\epsilon \log|\mathcal{S}_l|\right) && (3.58)\\ &\quad = n \sum_{\alpha=l}^{L} f_\alpha H(S_\alpha) - \sum_{\alpha=l}^{L} f_\alpha \left(1 + n\epsilon \log|\mathcal{S}_\alpha|\right) && (3.59) \end{aligned}$$

where (3.58) follows from the induction assumption (3.49). Finally, by the inequality chain (2.20) we have

$$\sum_{U\in\Omega_L^{(l-1)}} c_{l-1}(U)H(X_0,X_U|\mathbf{S}_0^{l-1}) \geq \sum_{U\in\Omega_L^{(l)}} c_l(U)H(X_0,X_U|\mathbf{S}_0^{l-1}) \tag{3.60}$$

$$\geq n\sum_{\alpha=l}^{L} f_\alpha H(S_\alpha) - \sum_{\alpha=l}^{L} f_\alpha\left(1+n\epsilon\log|\mathcal{S}_\alpha|\right) \tag{3.61}$$

This proves that the inequality (3.48) also holds for $m = l-1$.

Substituting (3.48) into (3.47) and dividing both sides of the inequality by n, we have

$$f_m R_0 + \sum_{l=1}^{L}\lambda_l R_l \geq f_m\sum_{\alpha=0}^{m} H(S_\alpha) + \sum_{\alpha=m+1}^{L} f_\alpha H(S_\alpha) - \delta_m(n,\epsilon) \tag{3.62}$$

where

$$\delta_m(n,\epsilon) = n^{-1}\sum_{\alpha=m}^{L} f_\alpha + \epsilon\left(f_m\sum_{\alpha=0}^{m}\log|\mathcal{S}_\alpha| + \sum_{\alpha=m+1}^{L} f_\alpha\log|\mathcal{S}_\alpha| + f_1 + f_m\right). \tag{3.63}$$

Let $n\to\infty$ and $\epsilon\to 0$. Note that $\delta_m(n,\epsilon)\to 0$ for any nonnegative $(\lambda_1,\ldots,\lambda_L)$ in $\mathbb{R}^L$ and any $m = 1,\ldots,L$. We thus have from (3.62) that

$$f_m R_0 + \sum_{l=1}^{L}\lambda_l R_l \geq f_m\sum_{\alpha=0}^{m} H(S_\alpha) + \sum_{\alpha=m+1}^{L} f_\alpha H(S_\alpha) \tag{3.64}$$

for any admissible rate tuple $(R_0, R_1, \ldots, R_L)$. This proves that $\mathcal{R}\subseteq\mathcal{R}_{sup}$ and hence completes the proof of Theorem 1.

CHAPTER IV

DISCUSSIONS

As shown in Section III, our proof of the optimality of superposition coding for the entire admissible rate region relies on a characterization of the superposition coding rate region which involves solving a sequence of linear programs (2.9) for $\alpha = 1, \dots, L$. For a general nonnegative $(\lambda_1, \dots, \lambda_L)$ in $\mathbb{R}^L$, the optimal solution of the linear program (2.9) cannot be written in closed-form for $\alpha = 2, \dots, L-1$. For the symmetrical situation where $\lambda_1 = \cdots = \lambda_L$, however, it is straightforward to verify that

$$c_\alpha(U) = \frac{\lambda_1}{\binom{L-1}{\alpha-1}}, \quad \forall U \in \Omega_L^{(\alpha)} \tag{4.1}$$

is an optimal solution to the linear program (2.9), giving the optimal value

$$f_\alpha = \binom{L}{\alpha} \frac{\lambda_1}{\binom{L-1}{\alpha-1}} = \frac{L\lambda_1}{\alpha} \tag{4.2}$$

for all $\alpha = 1, \dots, L$. In this section, we discuss some ramifications of (4.1) and (4.2).

A. An Explicit Characterization of the (R_0, R_s) Admissible Rate Region

First, let us use (4.2) to establish an *explicit* characterization of all possible tradeoffs between the encoding rate R_0 of the all-access encoder and the sum encoding rate R_s of the randomly accessible encoders, as captured by the (R_0, R_s) admissible rate region of the problem.

Formally, the (R_0, R_s) admissible rate region $\mathcal{R}'$ is defined as

$$\mathcal{R}' := \left\{ (R_0, R_s) : R_s = \sum_{l=1}^{L} R_l,\ (R_0, R_1, \dots, R_L) \in \mathcal{R} \right\} \tag{4.3}$$

where $\mathcal{R}$ is the admissible rate region of the problem. Let $\mathcal{R}^\dagger$ be the collection of all

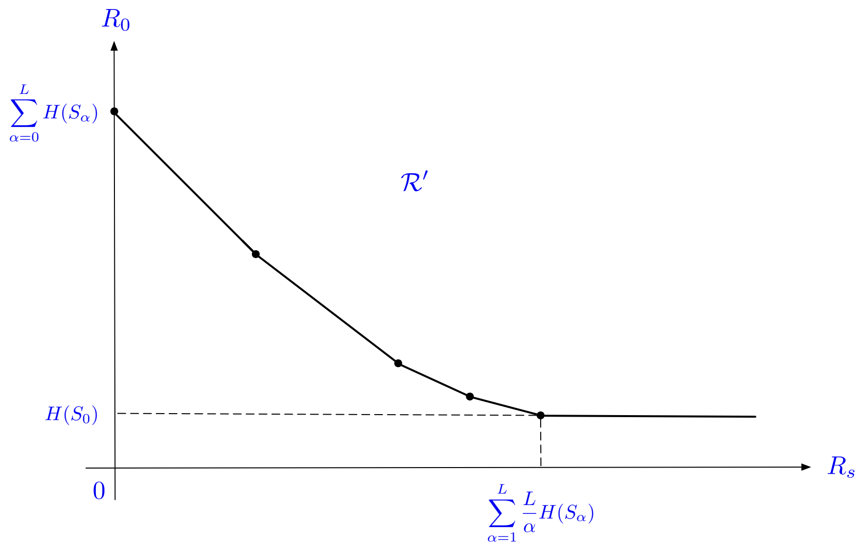

Fig. 2. An illustration of the (R_0, R_s) admissible rate region $\mathcal{R}'$. In general, the region is a (two-dimensional) polyhedron with polyhedral cone being the first quadrant and $L + 2$ faces.

nonnegative rate pairs (R_0, R_s) satisfying

$$R_0 + \frac{m}{L} R_s \geq \sum_{\alpha=0}^{m} H(S_\alpha) + \sum_{\alpha=m+1}^{L} \frac{m}{\alpha} H(S_\alpha) \tag{4.4}$$

for all $m = 0, 1, \ldots, L$. We have the following explicit characterization of the (R_0, R_s) admissible rate region $\mathcal{R}'$ (see Figure 2 for an illustration).

Theorem 2.

$$\mathcal{R}' = \mathcal{R}^\dagger. \tag{4.5}$$

Proof. By Theorem 1 and Proposition 2, the admissible rate region $\mathcal{R} = \mathcal{R}_{sup} = \mathcal{R}^*$, so the (R_0, R_s) admissible rate region $\mathcal{R}'$ can be written as

$$\mathcal{R}' := \left\{ (R_0, R_s) : R_s = \sum_{l=1}^{L} R_l, \; (R_0, R_1, \ldots, R_L) \in \mathcal{R}^* \right\}. \tag{4.6}$$

To show that $\mathcal{R}' \subseteq \mathcal{R}^\dagger$, let $(R_0, R_1, \ldots, R_L)$ be a nonnegative rate tuple in $\mathcal{R}^*$. By definition, $(R_0, R_1, \ldots, R_L)$ must satisfy (2.10) and (2.11) for all $m = 1, \ldots, L$ and all nonnegative $(\lambda_1, \ldots, \lambda_L)$ in $\mathbb{R}^L$. Let $R_s = \sum_{l=1}^{L} R_l$ and by (2.10), (R_0, R_s) must satisfy (4.4) for $m = 0$. Furthermore, let $\lambda_1 = \cdots = \lambda_L > 0$ and we have from (2.11) and (4.2) that (R_0, R_s) must also satisfy

$$\frac{L\lambda_1}{m} R_0 + \lambda_1 R_s \geq \frac{L\lambda_1}{m} \sum_{l=0}^{m} H(S_\alpha) + \sum_{l=m+1}^{L} \frac{L\lambda_1}{\alpha} H(S_\alpha) \tag{4.7}$$

for all $m = 1, \ldots, L$. Dividing both sides of the inequality by $L\lambda_1/m$ gives exactly the inequality (4.4). We have thus proved that $(R_0, R_s) \in \mathcal{R}^\dagger$ and hence $\mathcal{R}' \subseteq \mathcal{R}^\dagger$.

To show the reverse relationship $\mathcal{R}^\dagger \subseteq \mathcal{R}'$, let us show that for any $(R_0, R_s) \in \mathcal{R}^\dagger$, the rate tuple

$$(R_0, R_1, \ldots, R_L) = (R_0, R_s/L, \ldots, R_s/L) \tag{4.8}$$

must be in $\mathcal{R}^*$. The fact that $R_0 \geq H(S_0)$ follows directly from the inequality (4.4) with $m = 0$. Furthermore, for any nonnegative $(\lambda_1, \ldots, \lambda_L)$ in $\mathbb{R}^L$ and any $m = 1, \ldots, L$ we have

$$\begin{aligned}
f_m R_0 + \sum_{l=1}^{L} \lambda_l R_l &= f_m R_0 + \sum_{l=1}^{L} \lambda_l (R_s/L) && (4.9)\\
&= f_m R_0 + \frac{R_s}{L} \sum_{l=1}^{L} c_1(\{l\}) && (4.10)\\
&= f_m R_0 + \frac{f_1}{L} R_s && (4.11)\\
&\geq f_m R_0 + \frac{m f_m}{L} R_s && (4.12)\\
&= f_m \left(R_0 + \frac{m}{L} R_s \right) && (4.13)\\
&\geq f_m \left(\sum_{\alpha=0}^{m} H(S_\alpha) + \sum_{\alpha=m+1}^{L} \frac{m}{\alpha} H(S_\alpha) \right) && (4.14)\\
&= f_m \sum_{\alpha=0}^{m} H(S_\alpha) + \sum_{\alpha=m+1}^{L} \frac{m f_m}{\alpha} H(S_\alpha) && (4.15)\\
&\geq f_m \sum_{\alpha=0}^{m} H(S_\alpha) + \sum_{\alpha=m+1}^{L} f_\alpha H(S_\alpha) && (4.16)
\end{aligned}$$

where (4.12) and (4.16) are due to the inequality chain (2.17) so we have $f_1 \geq mf_m \geq \alpha f_\alpha$ for all $\alpha = m+1, \ldots, L$, and (4.14) follows from the fact that $(R_0, R_s) \in \mathcal{R}^\dagger$ so

$$R_0 + \frac{m}{L} R_s \geq \sum_{\alpha=0}^{m} H(S_\alpha) + \sum_{\alpha=m+1}^{L} \frac{m}{\alpha} H(S_\alpha). \tag{4.17}$$

We have thus proved that the rate tuple (4.8) must be in $\mathcal{R}^*$ and hence $\mathcal{R}^\dagger \subseteq \mathcal{R}'$.

Combining the facts that $\mathcal{R}' \subseteq \mathcal{R}^\dagger$ and $\mathcal{R}^\dagger \subseteq \mathcal{R}'$ gives $\mathcal{R}' = \mathcal{R}^\dagger$, which completes the proof of the theorem. □

B. A Sliding-Window Subset Entropy Inequality

By Theorem 2, the boundary of the (R_0, R_s) admissible rate region $\mathcal{R}'$ is determined by the supporting hyperplanes of the admissible rate region $\mathcal{R}$ with $\lambda_0 = 1$ and $\lambda_1 = \cdots = \lambda_L = m/L$ for $m = 0, 1, \ldots, L$. Therefore, to prove the optimality of superposition coding for the (R_0, R_s) admissible rate region $\mathcal{R}'$, one may only need to invoke the generalized Han's subset inequality for the special case where $\lambda_1 = \cdots = \lambda_L > 0$. Substituting (4.1) into the inequality chain (2.15) gives

$$\frac{\lambda_1}{\binom{L-1}{0}} \sum_{U \in \Omega_L^{(1)}} H(X_U) \geq \frac{\lambda_1}{\binom{L-1}{1}} \sum_{U \in \Omega_L^{(2)}} H(X_U) \geq \cdots \geq \frac{\lambda_1}{\binom{L-1}{L-1}} \sum_{U \in \Omega_L^{(L)}} H(X_U). \tag{4.18}$$

Dividing each term by $\lambda_1 L$ and using the fact that

$$\frac{1}{L\binom{L-1}{\alpha-1}} = \frac{1}{\alpha\binom{L}{\alpha}} \tag{4.19}$$

we have

$$\frac{1}{\binom{L}{1}} \sum_{U \in \Omega_L^{(1)}} H(X_U) \geq \frac{1}{\binom{L}{2}} \sum_{U \in \Omega_L^{(2)}} \frac{H(X_U)}{2} \geq \cdots \geq \frac{1}{\binom{L}{L}} \sum_{U \in \Omega_L^{(L)}} \frac{H(X_U)}{L} \tag{4.20}$$

which is precisely the classical subset entropy inequality of Han. We thus conclude that Han's subset inequality is sufficient to prove the optimality of superposition

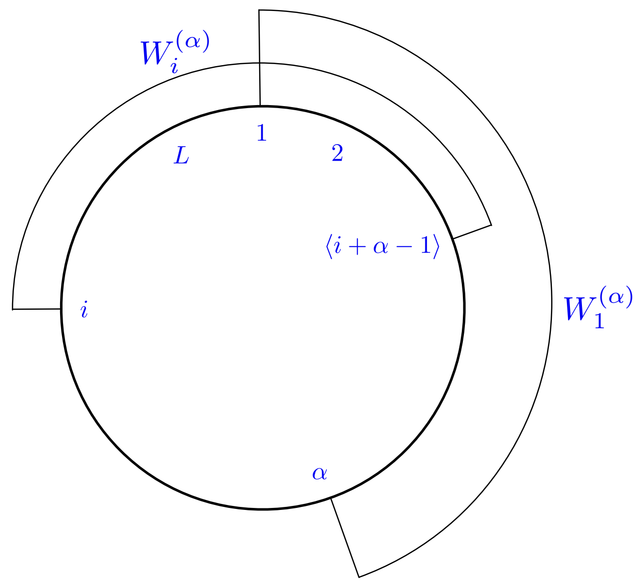

Fig. 3. An illustration of the sliding windows of length α when the integers $1, \ldots, L$ are circularly placed (clockwise) based on their natural order.

coding for the (R_0, R_s) admissible rate region.

Note, however, that when $\lambda_1 = \cdots = \lambda_L > 0$, the optimal solution to the linear program (2.9) is *not* unique for $\alpha = 2, \ldots, L-1$. Below we provide a different set of optimal solutions to the linear program (2.9) for $\alpha = 2, \ldots, L-1$, based on which we construct a new subset entropy inequality.

We shall start with the following notations. For any integer i, let us define

$$\langle i \rangle := \begin{cases} i \bmod L, & \text{if } i \bmod L \neq 0 \\ L, & \text{if } i \bmod L = 0 \end{cases} \tag{4.21}$$

and for any $i = 1, \ldots, L$ and $\alpha = 1, \ldots, L$, let

$$W_i^{(\alpha)} := \{i, \langle i+1 \rangle, \ldots, \langle i+\alpha-1 \rangle\}. \tag{4.22}$$

As illustrated in Figure 3, $W_i^{(\alpha)}$ represents a *sliding window* of length α starting with

i when the integers $1, \ldots, L$ are circularly placed (clockwise or counter clockwise) based on their natural order. Let $\mathcal{W}_L^{(\alpha)} = \{W_i^{(\alpha)} : i \in \Omega_L\}$ be the collection of the sliding windows of length α. It is straightforward to verify that when $\lambda_1 = \cdots = \lambda_L$, the following solution is also optimal for the linear program (2.9) for $\alpha = 2, \ldots, L-1$:

$$c_\alpha(U) = \begin{cases} \lambda_1/\alpha, & U \in \mathcal{W}_L^{(\alpha)} \\ 0, & U \in \Omega_L^{(\alpha)} \setminus \mathcal{W}_L^{(\alpha)}. \end{cases} \tag{4.23}$$

Substituting (4.23) into the inequality chain (2.15) suggests the following *sliding-window* subset entropy inequality, which we prove next.

Theorem 3 (A sliding-window subset entropy inequality)**.** *For any collection of L jointly distributed random variables $(X_1, \ldots, X_L)$, we have*

$$\sum_{i=1}^{L} H(X_{W_i^{(1)}}) \geq \frac{1}{2} \sum_{i=1}^{L} H(X_{W_i^{(2)}}) \geq \cdots \geq \frac{1}{L} \sum_{i=1}^{L} H(X_{W_i^{(L)}}). \tag{4.24}$$

The equalities hold when $X_1, \ldots, X_L$ are mutually independent of each other.

Proof. Note that it is sufficient to show that for any $\alpha = 2, \ldots, L$,

$$\frac{1}{\alpha - 1} \sum_{i=1}^{L} H(X_{W_i^{(\alpha-1)}}) \geq \frac{1}{\alpha} \sum_{i=1}^{L} H(X_{W_i^{(\alpha)}}). \tag{4.25}$$

Next, we shall prove (4.25) via an induction argument and the submodularity of the entropy function.

First, when $\alpha = 2$, we have

$$\begin{aligned}
\sum_{i=1}^{L} H(X_{W_i^{(1)}}) &= \sum_{i=1}^{L} H(X_i) && (4.26)\\
&= \frac{1}{2}\sum_{i=1}^{L} \big[H(X_i) + H(X_{\langle i+1\rangle})\big] && (4.27)\\
&\geq \frac{1}{2}\sum_{i=1}^{L} H(X_i, X_{\langle i+1\rangle}) && (4.28)\\
&= \frac{1}{2}\sum_{i=1}^{L} H(X_{W_i^{(2)}}) && (4.29)
\end{aligned}$$

where (4.28) follows from the independence bound on entropy. Thus, the inequality (4.25) holds for $\alpha = 2$.

Next, assume that the inequality (4.25) holds for $\alpha = r-1$. We have

$$\begin{aligned}
\sum_{i=1}^{L} H(X_{W_i^{(r-1)}}) &= \frac{1}{2}\sum_{i=1}^{L} \left[H(X_{W_i^{(r-1)}}) + H(X_{W_{\langle i+1\rangle}^{(r-1)}})\right] && (4.30)\\
&\geq \frac{1}{2}\sum_{i=1}^{L} \left[H(X_{W_i^{(r)}}) + H(X_{W_{\langle i+1\rangle}^{(r-2)}})\right] && (4.31)\\
&= \frac{1}{2}\sum_{i=1}^{L} H(X_{W_i^{(r)}}) + \frac{1}{2}\sum_{i=1}^{L} H(X_{W_{\langle i+1\rangle}^{(r-2)}}) && (4.32)\\
&= \frac{1}{2}\sum_{i=1}^{L} H(X_{W_i^{(r)}}) + \frac{1}{2}\sum_{i=1}^{L} H(X_{W_i^{(r-2)}}) && (4.33)\\
&\geq \frac{1}{2}\sum_{i=1}^{L} H(X_{W_i^{(r)}}) + \frac{1}{2}\cdot\frac{r-2}{r-1}\sum_{i=1}^{L} H(X_{W_i^{(r-1)}}) && (4.34)
\end{aligned}$$

where (4.31) follows from the submodularity of entropy [12, Ch. 14.A]

$$H(X_U) + H(X_V) \geq H(X_{U\cup V}) + H(X_{U\cap V}) \qquad (4.35)$$

for $U = W_i^{(r-1)}$ and $V = W_{\langle i+1\rangle}^{(r-1)}$ so $U \cup V = W_i^{(r)}$ and $U \cap V = W_{\langle i+1\rangle}^{(r-2)}$, and (4.34) follows from the induction assumption

$$\frac{1}{r-2}\sum_{i=1}^{L} H(X_{W_i^{(r-2)}}) \geq \frac{1}{r-1}\sum_{i=1}^{L} H(X_{W_i^{(r-1)}}). \tag{4.36}$$

Moving the second term on the right-hand side of (4.34) to the left and multiplying both sides by $2/r$ gives

$$\frac{1}{r-1}\sum_{i=1}^{L} H(X_{W_i^{(r-1)}}) \geq \frac{1}{r}\sum_{i=1}^{L} H(X_{W_i^{(r)}}). \tag{4.37}$$

We have thus proved that the inequality (4.25) also holds for $\alpha = r$.

Finally, note that when $X_1, \ldots, X_L$ are mutually independent, we have

$$\frac{1}{\alpha}\sum_{i=1}^{L} H(X_{W_i^{(\alpha)}}) = \sum_{i=1}^{L} H(X_i), \quad \forall \alpha = 1, \ldots, L. \tag{4.38}$$

This completes the proof of Theorem 3. □

Just like the classical subset entropy inequality of Han, the sliding-window subset entropy inequality is also sufficient to prove the optimality of superposition coding for the (R_0, R_s) admissible rate region. In fact, Han's subset inequality can be derived from the sliding-window subset entropy inequality via a very simple permutation argument as follows.

Let π be a *permutation* on Ω_L. For any $i = 1, \ldots, L$ and $\alpha = 1, \ldots, L$, let

$$W_{\pi,i}^{(\alpha)} := \{\pi^{-1}(i), \pi^{-1}(\langle i+1\rangle), \ldots, \pi^{-1}(\langle i+\alpha-1\rangle)\}. \tag{4.39}$$

By Theorem 3, we have

$$\sum_{i=1}^{L} H(X_{W_{\pi,i}^{(1)}}) \geq \frac{1}{2}\sum_{i=1}^{L} H(X_{W_{\pi,i}^{(2)}}) \geq \cdots \geq \frac{1}{L}\sum_{i=1}^{L} H(X_{W_{\pi,i}^{(L)}}). \tag{4.40}$$

Averaging (4.40) over all possible permutations π, we have

$$\frac{1}{L!}\sum_{\pi}\left[\sum_{i=1}^{L} H(X_{W_{\pi,i}^{(1)}})\right] \geq \frac{1}{L!}\sum_{\pi}\left[\frac{1}{2}\sum_{i=1}^{L} H(X_{W_{\pi,i}^{(2)}})\right] \geq \cdots \geq \frac{1}{L!}\sum_{\pi}\left[\frac{1}{L}\sum_{i=1}^{L} H(X_{W_{\pi,i}^{(L)}})\right]. \tag{4.41}$$

Note that for any $\alpha = 1, \ldots, L$,

$$\sum_{\pi}\sum_{i=1}^{L} H(X_{W_{\pi,i}^{(\alpha)}}) = L \cdot \alpha!(L-\alpha)! \sum_{U \in \Omega_L^{(\alpha)}} H(X_U). \tag{4.42}$$

Substituting (4.42) into (4.41) and dividing each term by L give the classical subset entropy inequality of Han (4.20).

CHAPTER V

CONCLUSION

This thesis considered the problem of SMDC where, in addition to the randomly accessible encoders, there is also an *all-access* encoder. This is a natural extension of the original SMDC problem introduced by Roche [7] and Yeung [10], for which superposition coding was shown to be optimal in terms of achieving the minimum sum rate [8] and the entire admissible rate region [11]. For this generalized setting, it was shown that superposition coding remains optimal in terms of achieving the entire admissible rate region of the problem. Key to our proof is to identify the supporting hyperplanes that define the boundary of the admissible rate region and then build on the result of Yeung and Zhang [11] on a generalization of Han's subset inequality. As a special case, the (R_0, R_s) admissible rate region, which captures all possible tradeoffs between the encoding rate R_0 of the all-access encoder and the sum encoding rate R_s of the randomly accessible encoders, is explicitly characterized. To provide an explicit proof of the optimality of superposition coding, a new sliding-window subset entropy inequality is introduced and is shown to directly imply the classical subset entropy inequality of Han. It is our hope that the sliding-window subset entropy inequality will be of interest to some other network compression and communication problems as well.

As a side development, a more structured proof of the generalized Han's subset inequality is provided, which completely separates the entropy argument from the analysis of the underlying linear programs. The proof is based on a subset entropy inequality recently established by Madiman and Tetali [3] and an analysis result on the underlying linear programs that we lift from the original proof of Yeung and Zhang [11]. We believe that this improved proof represents a better reflection on the technical nature of the generalized Han's subset inequality.

Finally, we mention here that the more general setting, wherein, the encoders are completely ordered, was studied in the recent work [4]. In this "asymmetrical" setting, as demonstrated in [4], coding across different sources is generally needed to achieve the entire rate region of the problem.

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APPENDIX A

PROOF OF LEMMA 2

Without loss of generality, let us assume that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L. \tag{A.1}$$

When $L = 2$, the optimal solutions to the linear program (2.9) are unique and are given by

$$c_1(\{l\}) = \lambda_l, \quad l = 1, 2 \tag{A.2}$$

and

$$c_2(\{1, 2\}) = \lambda_2. \tag{A.3}$$

When $f_2 = \lambda_2 > 0$, it is straightforward to verify that

$$g_{\{1,2\}}(\{l\}) = \lambda_l / \lambda_2, \quad l = 1, 2 \tag{A.4}$$

is a fractional cover of $(\{1, 2\}, \mathcal{V}_{\{1,2\}})$ and satisfies (3.28).

Assume that a set of desired fractional covers $\mathcal{G}_L^{(\alpha)}$ exists for some $L = N - 1$ whenever $f_\alpha > 0$. To show that the desired fractional covers also exist for $L = N$ when $f_\alpha > 0$, we shall consider the following three cases separately. In each of the cases below, we will construct a set of fractional covers $\mathcal{G}_N^{(\alpha)} = \{g_U : U \in \Omega_N^{(\alpha)}\}$ which leads to the *same* $c_{\alpha-1}$ as those constructed in the proof of [11, Theorem 3]. Therefore, the readers are referred to the proof of [11, Theorem 3] for the optimality of such $c_{\alpha-1}$. Instead, our focus is on proving that the so-constructed g_U is indeed a fractional cover of $(U, \mathcal{V}_U)$ for each $U \in \Omega_N^{(\alpha)}$.

Case 1: $\lambda_1 \leq \frac{\lambda_2+\cdots+\lambda_N}{\alpha-1}$. In this case, let us consider for any $U \in \Omega_N^{(\alpha)}$

$$g_U(V) = \frac{1}{\alpha-1}, \quad \forall V \in \mathcal{V}_U \tag{A.5}$$

so

$$c_{\alpha-1}(V) = \sum_{U \in \Omega_N^{(\alpha)}, U \supseteq V} \frac{c_\alpha(U)}{\alpha-1}, \quad \forall V \in \Omega_N^{(\alpha-1)}. \tag{A.6}$$

Note that this gives the same $c_{\alpha-1}$ as [11, Eq. (39)] so it is an optimal solution to the linear program (3.24). Further note that for any $i \in U$, $|\{V \in \mathcal{V}_U : V \ni i\}| = \alpha - 1$ so

$$\sum_{V \in \mathcal{V}_U, V \ni i} g(V) = 1. \tag{A.7}$$

We thus conclude that g_U is a fractional cover of $(U, \mathcal{V}_U)$ for any $U \in \Omega_N^{(\alpha)}$. This completes the proof of Case 1.

Case 2: $\lambda_1 > \frac{\lambda_2+\cdots+\lambda_N}{\alpha-2}$. In this case, by [11, Lemma 6] for any c_α which is an optimal solution to the linear program (2.9), $c_\alpha(U) > 0$ implies that $U \ni 1$. Furthermore, by [11, Lemma 8] $\tilde{c}_{\alpha-1} = \{\tilde{c}_{\alpha-1}(\tilde{U}) : \tilde{U} \subseteq \tilde{\Omega}_{N-1} := \{2, \ldots, N\}\}$ where

$$\tilde{c}_{\alpha-1}(\tilde{U}) = c_\alpha(\{1\} \cup \tilde{U}) \tag{A.8}$$

is an optimal solution to the linear program

$$\begin{array}{rl} \max & \sum_{\tilde{U} \in \tilde{\Omega}_{N-1}^{(\alpha-1)}} \tilde{c}_{\alpha-1}(\tilde{U}) \\ \text{subject to} & \sum_{\tilde{U} \in \tilde{\Omega}_{N-1}^{(\alpha-1)}, \tilde{U} \ni l} \tilde{c}_{\alpha-1}(\tilde{U}) \leq \lambda_l, \quad \forall l = 2, \ldots, N \\ & \tilde{c}_{\alpha-1}(\tilde{U}) \geq 0, \quad \forall \tilde{U} \in \tilde{\Omega}_{N-1}^{(\alpha-1)} \end{array} \tag{A.9}$$

with the optimal solution $\tilde{f}_{\alpha-1} = f_\alpha > 0$. Thus, by the induction assumption there exists a set of fractional covers $\tilde{\mathcal{G}}_{N-1}^{(\alpha-1)} = \{\tilde{g}_{\tilde{U}} : \tilde{U} \in \tilde{\Omega}_{N-1}^{(\alpha-1)}\}$ such that $\tilde{c}_{\alpha-2} = \{\tilde{c}_{\alpha-2}(\tilde{V}) :$

$\tilde{V} \in \tilde{\Omega}_{N-1}^{(\alpha-2)}\}$ where

$$\tilde{c}_{\alpha-2}(\tilde{V}) = \sum_{\tilde{U} \in \tilde{\Omega}_{N-1}^{(\alpha-1)}, \tilde{U} \supseteq \tilde{V}} \tilde{c}_{\alpha-1}(\tilde{U}) \tilde{g}_{\tilde{U}}(\tilde{V}) \tag{A.10}$$

is an optimal solution to the linear program

$$\begin{array}{rl} \max & \sum_{\tilde{V} \in \tilde{\Omega}_{N-1}^{(\alpha-2)}} \tilde{c}_{\alpha-2}(\tilde{V}) \\ \text{subject to} & \sum_{\tilde{V} \in \tilde{\Omega}_{N-1}^{(\alpha-2)}, \tilde{V} \ni l} \tilde{c}_{\alpha-2}(\tilde{V}) \leq \lambda_l, \quad \forall l = 2, \ldots, N \\ & \tilde{c}_{\alpha-2}(\tilde{V}) \geq 0, \quad \forall \tilde{V} \in \tilde{\Omega}_{N-1}^{(\alpha-2)}. \end{array} \tag{A.11}$$

Consider for any $U \in \Omega_N^{(\alpha)}$ such that $1 \in U$

$$g_U(V) := \begin{cases} \tilde{g}_{\tilde{U}}(\tilde{V}), & \text{if } V = \{1\} \cup \tilde{V} \text{ for some } \tilde{V} \in \tilde{\mathcal{V}}_{\tilde{U}} \\ 0, & \text{otherwise} \end{cases} \tag{A.12}$$

where $\tilde{U} = U \setminus \{1\}$. For any $U \in \Omega_N^{(\alpha)}$ such that $1 \notin U$, we can pick g_U to be *any* fractional cover of $(U, \mathcal{V}_U)$ as $c_\alpha(U) = 0$. Note that for any $V \in \Omega_N^{(\alpha-1)}$ such that $1 \in V$

$$\begin{aligned} c_{\alpha-1}(V) &= \sum_{U \in \Omega_N^{(\alpha)}, U \supseteq V} c_\alpha(U) g_U(V) && \text{(A.13)} \\ &= \sum_{\tilde{U} \in \tilde{\Omega}_{N-1}^{(\alpha-1)}, \tilde{U} \supseteq \tilde{V}} c_\alpha(\{1\} \cup \tilde{U}) \tilde{g}_{\tilde{U}}(\tilde{V}) && \text{(A.14)} \\ &= \sum_{\tilde{U} \in \tilde{\Omega}_{N-1}^{(\alpha-1)}, \tilde{U} \supseteq \tilde{V}} \tilde{c}_{\alpha-1}(\tilde{U}) \tilde{g}_{\tilde{U}}(\tilde{V}) && \text{(A.15)} \\ &= \tilde{c}_{\alpha-2}(\tilde{V}) && \text{(A.16)} \end{aligned}$$

where $\tilde{V} = V \setminus \{1\}$, and for any $V \in \Omega_N^{(\alpha-1)}$ such that $1 \notin V$

$$c_{\alpha-1}(V) = \sum_{U \in \Omega_N^{(\alpha)}, U \supseteq V} c_\alpha(U) g_U(V) = 0. \tag{A.17}$$

This gives the same $c_{\alpha-1}$ as [11, Eq. (46)] so it is an optimal solution to the linear program (3.24). It remains to show that g_U is a fractional cover of $(U, \mathcal{V}_U)$ for any $U \in \Omega_N^{(\alpha)}$ such that $U \ni 1$.

For any $i \in U \setminus \{1\}$, we have

$$\sum_{V \in \mathcal{V}_U, V \ni i} g_U(V) = \sum_{V \in \mathcal{V}_U, V \supseteq \{1,i\}} g_U(V) = \sum_{\tilde{V} \in \tilde{\mathcal{V}}_{\tilde{U}}, \tilde{V} \ni i} \tilde{g}_{\tilde{U}}(\tilde{V}) \geq 1 \tag{A.18}$$

and

$$\sum_{V \in \mathcal{V}_U, V \ni 1} g_U(V) \geq \sum_{V \in \mathcal{V}_U, V \supseteq \{1,i\}} g_U(V) \geq 1. \tag{A.19}$$

This completes the proof of Case 2.

Case 3: $\frac{\lambda_2 + \cdots + \lambda_N}{\alpha - 1} < \lambda_1 \leq \frac{\lambda_2 + \cdots + \lambda_N}{\alpha - 2}$. In this case, we shall need the following notations. For any $U \in \Omega_N^{(\alpha)}$ and any integer τ between 1 and α, denote by $a_U(\tau)$ the *smallest* positive integer l such that

$$|\{1, \ldots, l\} \cap U| = \tau. \tag{A.20}$$

Let

$$W_\tau(U) := U \setminus \{a_U(\tau)\} \tag{A.21}$$

so $W_\tau(U) \in \Omega_N^{(\alpha-1)}$ for all $U \in \Omega_N^{(\alpha)}$ and all integer τ between 1 and α. Define a mapping $\xi_{U,m,\tau} : \Omega_N^{(\alpha-1)} \to \mathbb{R}^+$ for each $U \in \Omega_N^{(\alpha)}$, each integer m between 2 and α, and each integer τ between m and α by

$$\xi_{U,m,\tau}(V) := \begin{cases} \frac{b_{m-1}^{(\alpha)} - b_m^{(\alpha)}}{f_\alpha}, & \text{if } V = W_\tau(U) \\ 0, & \text{otherwise} \end{cases} \tag{A.22}$$

where

$$\begin{aligned} b_l^{(\alpha)} &:= \lambda_l - \tilde{\lambda}_l & \text{(A.23)} \\ \text{and} \quad \tilde{\lambda}_l &:= \sum_{U \in \Omega_N^{(\alpha)}, U \ni l} c_\alpha(U), \quad \forall l = 1, \ldots, L. & \text{(A.24)} \end{aligned}$$

Let

$$\beta := \sum_{m=2}^{\alpha-1} (b_1^{(\alpha)} - b_m^{(\alpha)}) \tag{A.25}$$

and consider the set of fractional covers $\mathcal{G}_N^{(\alpha)} = \{g_U : U \in \Omega_N^{(\alpha)}\}$ where

$$g_U(V) = \left(1 - \frac{\beta}{f_\alpha}\right) \frac{1}{\alpha - 1} + \sum_{m=2}^{\alpha} \sum_{\tau=m}^{\alpha} \xi_{U,m,\tau}(V), \quad \forall V \in \mathcal{V}_U. \tag{A.26}$$

This gives

$$c_{\alpha-1}(V) = \left(1 - \frac{\beta}{f_\alpha}\right) \sum_{U \in \Omega_N^{(\alpha)}, U \supseteq V} \frac{c_\alpha(U)}{\alpha - 1} + \sum_{U \in \Omega_N^{(\alpha)}, U \supseteq V} \sum_{m=2}^{\alpha} \sum_{\tau=m}^{\alpha} \xi_{U,m,\tau}(V) c_\alpha(U), \quad \forall V \in \Omega_N^{(\alpha-1)} \tag{A.27}$$

which is the same as [11, Eq. (55)] and hence is an optimal solution to the linear program (3.24). It remains to show that g_U is a fractional cover of $(U, \mathcal{V}_U)$ for any $U \in \Omega_N^{(\alpha)}$

Note that for any $i \in U$,

$$\sum_{V \in \mathcal{V}_U, V \ni i} \left(1 - \frac{\beta}{f_\alpha}\right) \frac{1}{\alpha - 1} = 1 - \frac{\beta}{f_\alpha} \tag{A.28}$$

and

$$\begin{aligned}
\sum_{V \in \mathcal{V}_U, V \ni i} \sum_{m=2}^{\alpha} \sum_{\tau=m}^{\alpha} \xi_{U,m,\tau}(V) &= \sum_{m=2}^{\alpha} \sum_{\tau=m}^{\alpha} \left(\sum_{V \in \mathcal{V}_U, V \ni i} \xi_{U,m,\tau}(V) \right) && \text{(A.29)} \\
&= \sum_{m=2}^{\alpha} \sum_{\tau=m}^{\alpha} \frac{b_{m-1}^{(\alpha)} - b_m^{(\alpha)}}{f_\alpha} 1_{\{a_U(\tau) \neq i\}} && \text{(A.30)} \\
&= \sum_{m=2}^{\alpha} \frac{b_{m-1}^{(\alpha)} - b_m^{(\alpha)}}{f_\alpha} \left(\sum_{\tau=m}^{\alpha} 1_{\{a_U(\tau) \neq i\}} \right) && \text{(A.31)} \\
&\geq \sum_{m=2}^{\alpha} \frac{b_{m-1}^{(\alpha)} - b_m^{(\alpha)}}{f_\alpha} (\alpha - m) && \text{(A.32)} \\
&= \frac{\beta}{f_\alpha} && \text{(A.33)}
\end{aligned}$$

where (A.33) follows from [11, Eq. (66)]. Combing (A.28) and (A.33) gives

$$\sum_{V\in\mathcal{V}_U, V\ni i} g_U(V) \geq 1 - \frac{\beta}{f_\alpha} + \frac{\beta}{f_\alpha} = 1. \tag{A.34}$$

We thus conclude that g_U as defined in (A.26) is indeed a fractional cover of $(U, \mathcal{V}_U)$ for any $U \in \Omega_N^{(\alpha)}$. This completes the proof of Case 3.

VITA

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